MLS-based variable-node elements compatible with quadratic interpolation. Part I: Formulation and application for non-matching meshes

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SUMMARY

Two-dimensional variable-node elements compatible with quadratic interpolation are developed using the moving least-square (MLS) approximation. The mapping from the parental domain to the physical element domain is implicitly obtained from MLS approximation, with the shape functions and their derivatives calculated and saved only at the numerical integration points. It is shown that the present MLS-based variable-node elements meet the patch test if a sufficiently large number of integration points are employed for numerical integration. The cantilever problem with non-matching meshes is chosen to check the feasibility of the present MLS-based variable-node elements, and the result is compared with that from the lower-order case compatible with linear interpolation. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: MLS-based variable-node element; non-matching meshes; moving least-square approximation

1. INTRODUCTION

For numerical analysis of structures in the framework of finite element method (FEM), the mesh construction is unavoidable, and meshing itself may not be straightforward, particularly if the configuration of structure is complex. Moreover, it could be costly and laborious to model and handle a discontinuity, which exists inherently in the structure, or sometimes caused by
modelling. Here we classify the discontinuities into two categories. One is the mesh discontinuity which is found in non-matching mesh. Co-operative partitioned modelling of an immense structure or modelling using different types of elements just for the convenience may lead to non-matching meshes. The other is the material or geometric discontinuity that implies the discontinuity of material properties, material responses or geometry itself. For example, the material or geometric discontinuity includes an interface of composite material or structures in contact, phase transformation from liquid to solid or vice versa, shear band and cracks in materials and structures.

In this paper, we focus on the mesh discontinuity and propose elements to deal with the non-matching meshes induced by the partitioned modelling. Up to now, several methods based on finite elements have been proposed for the treatment of non-matching interfaces. First of all, the modification of shape functions to impose a constraint for some nodes on slave boundaries have been used [1, 2]. The hybrid FEM, suggested in References [3–5], deals with the interface discontinuity via Lagrange multipliers. The mortar FEM in References [6–8] enforces the continuity condition in weak form by the orthogonality relation between the jump and the Lagrange multipliers space. Similar methods has also been proposed in References [9–13]. Interface Element Method (IEM) was introduced in References [14–16] and this method treats the non-matching finite element meshes via the moving least-square (MLS) approximation [17, 18]. Recently, Cho et al. [19] proposed MLS-based variable-node elements to treat the non-matching meshes for the case of linear interpolation. The present MLS-based variable-node elements have several advantages over the previous IEM approach [14–16], as addressed in Reference [19]. Firstly, they retain the conventional element concept of FEM. Secondly, the implementation of MLS-based variable-node elements is simple and straightforward, as it follows the implementation procedure of the conventional FEM other than that the shape functions are derived via MLS approximation. Lastly, MLS-based variable-node elements have potentials for diverse applications, which is manifested in Part II of this work [20]. In this paper, we extend the MLS-based variable-node elements to the case of a higher-order interpolation, and thus propose MLS-based variable-node elements compatible with quadratic interpolation. In Section 2, the underlying idea of the element to treat the non-matching meshes is suggested, and the MLS-based variable-node elements are proposed for the quadratic case. In Section 3, the patch test is performed for the present MLS-based variable-node element, and a cantilever beam problem is demonstrated for comparison to the result from the case compatible with linear interpolation [19]. In Section 4, this study is summarized and some concluding remarks are addressed.

2. MLS-BASED VARIABLE-NODE ELEMENTS COMPATIBLE WITH QUADRATIC ELEMENTS

For the case of non-matching meshes composed of quadratic elements, we can use the procedure similar to the linear case [19]. As in Figure 1(a), let us consider the non-matching interface in a mesh of quadratic elements. Firstly, the mid-point nodes represented by white-coloured circles in Figure 1(b), are removed. After that, nodes represented by squares are added to the neighbouring elements along the interface at the opposite position of the nodes remaining at the interface as in Figure 1(c). Contrary to the case of linear interpolation [19], the node-matching process does not guarantee the compatible interface because the shape function of a mid-node is
different, in its interpolation, from the shape function of a corner node (see Figure 2). Figure 2 is the shape functions of an one-dimensional quadratic element, which represent the shape functions of a two-dimensional quadratic element as well at the boundary of element. To attain the compatibility of the quadratic shape function along the interface, we add nodes, indicated by triangle in Figure 1(d), at the mid-points between the squares and the black circles. Then, the joining elements on the left and on the right share nodes along the interface. This node sharing occurs in the following way. Firstly, a mid-node, denoted by triangle, in Figure 1, on one side meets with a mid-node on the other side. In this case, the two joining nodes are of the same type so that the neighbouring elements can share the same shape function for the global node shared. Secondly, a corner node, indicated by solid circle, on one side meets with a new node, indicated by shaded square, on the other side. For compatibility in this circumstance, the node of shaded square should retain the same type of shape function as the corner node so that the two joining elements can share the same shape function for the global node shared on the interface. On the master element domain in Figure 3, this implies that node 9, corresponding to the node of shaded square, should be compatible with the corner node 1 or 4. It would not sound plausible to construct this element using the traditional shape function of the polynomial.
Figure 2. The configuration of quadratic shape functions for each node A, B, and C at the boundary of element.

Figure 3. The master element for 10-node quadratic element and node numbering.

type. To surmount the limitation of the polynomial-type interpolation, we rely upon MLS-based approximation, which enables us to extend the function space of the regular finite elements. For this, we first consider the following weight functions, \( W_i \) on the master element in Figure 3:

\[
\begin{align*}
W_1 &= 0 \quad \text{on } 2, 6, 3, 7, 4, 8, 9, 9, 10, 10, 11, 5, 2 \\
W_2 &= 0 \quad \text{on } 3, 7, 4, 8, 9, 9, 10, 10, 11, 5, 2 \\
W_3 &= 0 \quad \text{on } 4, 8, 9, 9, 10, 10, 11, 5, 2
\end{align*}
\]
where $W_i$ denotes the weight function associated with node $i$, and $ij$ represents the line segment between node $i$ and node $j$ in Figure 3. In this study, the weight functions of MLS approximation for the master element in Figure 3 are determined as in Figure 4. In Figure 4, a black-coloured circle represents the position of the node associated with the weight function, and a square the nodal position of a non-zero value of the weight function, while the greyed region means the area that has a non-zero value of the weight function. At the nodes of white-coloured circle, the values of the weight functions are zero. Figure 5 shows the weight function of Figure 4 with adoption of quartic spline. With the weight functions of Figure 5 and the second-order basis, $p = [1 \ x \ y \ x^2 \ y^2 \ xy]$, the shape functions of the present 10-node
Figure 5. The weight functions used in MLS approximation at each node with adoption of quartic spline for the case of Figure 3.

The 10-node element as in Figure 3 can be easily extended to other variable-node elements. For example, we can devise a 12-node element as in Figure 7 by adding two more nodes to the aforementioned 10-node element. However, the negative Jacobian may appear when the mapping from the natural co-ordinates to the physical co-ordinates is conducted using the shape function of Figure 7. Figure 8(b) is the contour plot of the negative Jacobian, when the mapping is carried out as in Figure 8(a) for the case of the 12-node element. Fortunately, we can eliminate this problem by modifying the master element as in Figure 9. Note that we have added an additional node at the centre of the parental domain. This 13-node element may be contemplated as being extended, with two more nodes added, from the 11-node element in Figure 10. Then, the weight functions are modified as in Figures 11 and 12, which now replace Figures 4 and 5. The detailed weight functions for Figure 10 is given in the following:

\[
\begin{align*}
W_1 &= W(\xi, -1, 1)W(\eta, -1, 0) \\
W_2 &= W(\xi, 1, -1)W(\eta, -1, 1) \\
W_3 &= W(\xi, 1, -1)W(\eta, 1, -1) \\
W_4 &= W(\xi, -1, 1)W(\eta, 1, 0) \\
W_5 &= W(\xi, 0, 1)W(\eta, -1, 1)
\end{align*}
\]
Figure 6. The shape functions obtained using MLS approximation with the weight function of Figure 5 at each node.

Figure 7. The master element for 12-node quadratic element and node numbering.
Figure 8. (a) Mapping from a 12-node master element to an element in physical co-ordinates; and (b) the Jacobian distribution for mapping (a).

Figure 9. The modified master element for 13-node quadratic element and node numbering.

\[ W_6 = W(\zeta, 1, -1)W(\eta, 0, -1) \]
\[ W_7 = W(\zeta, 0, 1)W(\eta, 1, -1) \]
\[ W_8 = W(\zeta, 0, 1)W(\eta, 0, 1) \]
\[ W_9 = W(\zeta, -1, 1)W(\eta, 0.5, 0) \]
\[ W_{10} = W(\zeta, -1, 1)W(\eta, 0, -1) \]
\[ W_{11} = W(\zeta, -1, 1)W(\eta, -0.5, -1) \]
Figure 10. The master element for 11-node quadratic element and node numbering.

Figure 11. The weight functions used in MLS approximation at each node for the case of Figure 10, (black-coloured circle: the position of node associated with each weight function, square: the node with a non-zero value of the associated weight function, white-coloured circle: the node with a zero value of the associated weight function, grey-coloured region: the region with a non-zero value of the associated weight function).
Figure 12. The weight functions used in MLS approximation at each node with adoption of quartic spline for the case of Figure 10.

where

\[ W(x, x_0, x_1) = (1 - 6\tilde{x}^2 + 8\tilde{x}^3 - 3\tilde{x}^4) \quad \text{for} \quad \tilde{x} = \frac{|x - x_0|}{x_1 - x_0} \leq 1 \]

\[ = 0 \quad \text{for} \quad \tilde{x} = \frac{|x - x_0|}{x_1 - x_0} \geq 1 \quad (3) \]

Then, the shape functions are calculated as in Figure 13 using MLS approximation. As seen in Figure 14, the negative Jacobian is eliminated if the shape functions of Figure 13 is adopted for the extreme case as given in Figure 8(a).

The above-mentioned MLS-based 11-node element, which is compatible with the quadratic element, can be extended to \((9 + 2n)\)-node element by using the following weight functions: \((n = 1, 2, 3, \ldots)\).

\[
W_1 = W(\xi, \xi_1, \xi_2)W(\eta, \eta_1, \eta_1 + 2\Delta) \\
W_2 = W(\xi, \xi_2, \xi_1)W(\eta, \eta_2, \eta_3) \\
W_3 = W(\xi, \xi_3, \xi_4)W(\eta, \eta_3, \eta_2) \\
W_4 = W(\xi, \xi_4, \xi_3)W(\eta, \eta_4, \eta_4 - 2\Delta)
\]
Figure 13. The shape functions obtained using MLS approximation with the weight function of Figure 12 at each node.

Figure 14. (a) Mapping from a 13-node master element to an element in physical co-ordinates; and (b) the Jacobian distribution for mapping (a).
Figure 15. An illustration of \((9 + 2n)\)-node element.

\[
\begin{align*}
W_5 &= W(\xi_5, \xi_7, \xi_2)W(\eta_5, \eta_7) \\
W_6 &= W(\xi_6, \xi_1, \xi_3)W(\eta_6, \eta_3) \\
W_7 &= W(\xi_7, \xi_4, \xi_5)W(\eta_7, \eta_5) \\
W_8 &= W(\xi_8, \xi_3, \xi_1)W(\eta_8, \eta_3) \\
W_9 &= W(\xi_9, \xi_1, \xi_1)W(\eta_9, \eta_1) \\
W_{9+1} &= W(\xi_9, \xi_9+1, \xi_3)W(\eta_9+1, \eta_9+1 - 2\Delta) \\
W_{9+2} &= W(\xi_9+1, \xi_9+2, \xi_3)W(\eta_9+1, \eta_9+1 - \Delta) \\
& \vdots \\
W_{9+2n-1} &= W(\xi_9+2n-1, \xi_9+2n-1, \xi_3)W(\eta_9+2n-1, \eta_9+2n-1 - 2\Delta) \\
W_{9+2n} &= W(\xi_9+2n, \xi_9+2n, \xi_3)W(\eta_9+2n, \eta_9+2n - \Delta)
\end{align*}
\]

(4)

where \(\xi_i\) and \(\eta_i\) denote the co-ordinate of \(\xi\) and \(\eta\) at the node \(i\) and \(\Delta = 2/2(n+1) = 1/(n+1)\) is the length of the equally divided segment illustrated in Figure 15.

3. NUMERICAL RESULTS

All numerical experiments in this section are conducted in the framework of two-dimensional plane-stress linear elasticity. The \(3 \times 3\) Gauss quadrature rule is used for the standard 8-node quadratic elements. For the MLS-based variable-node elements, a higher-order Gaussian
interpolation is required because their shape functions involve complex rational functions. For the 11-noded element, the element domain is divided into two, and the $6 \times 6$ Gauss quadrature is employed for each of the two domains (see Figure 16). As seen in Figure 16, this integration procedure can be generalized as $(6 \times 6) \times (n + 1)$ Gauss point integration for the general MLS-based $(9 + 2n)$-node elements.

3.1. Patch test

To check the validity of the proposed MLS-based element, we conduct a patch test employing an irregular mesh as shown in Figure 17 where $\sigma$ is 200 MPa. Figure 18 shows the contour plot of $\sigma_{xx}$ when $6 \times 6$ integrations are used for the MLS-based elements and Figure 19 shows the result of the patch test in terms of the relative error of energy norm for increasing order of Gaussian integration. The relative error of energy norm is defined as

$$r_e = \frac{\| \varepsilon_{\text{num}} - \varepsilon_{\text{exact}} \|_{\Omega}}{\| \varepsilon_{\text{exact}} \|_{\Omega}}$$

(5)

where

$$\| \varepsilon \|_{\Omega} = \left( \frac{1}{2} \int_{\Omega} \varepsilon^T D \varepsilon \, d\Omega \right)^{1/2}$$

(6)

$\Omega$ is the entire domain, $\varepsilon_{\text{num}}$ is the strain calculated numerically and $\varepsilon_{\text{exact}}$ is the exact strain in this patch test. From this, it is seen that the proposal MLS-based quadratic element passes the patch test as long as a sufficiently high order of Gaussian integration, say higher than $6 \times 6$, is chosen for numerical integration.
Figure 17. Patch-test model for quadratic-type MLS-based element under tension.

Figure 18. The contour plot of $\sigma_{xx}$ for Figure 17.

Next, to check the capability of representing a linear stress variation, we consider a beam is under pure bending like Figure 20. The stress $\sigma_{xx}$ increases linearly proportional to the $y$ co-ordinate in bending:

$$\sigma_{xx} = -\frac{M_y}{I}$$  \hfill (7)
where the moment employed in the model is $38400 \times 10^6$ Nm, $I = D^3/12$ and $D = 12$ m is height. Therefore, the pure bending example provides a test for checking the accuracy of the quadratic-type element. The contour plot of $\sigma_{xx}$ is shown in Figure 21 and the relative error of energy norm is depicted in Figure 22 as the order of Gaussian integration in the
Figure 21. The contour plot of $\sigma_{xx}$ for Figure 20.

Figure 22. The convergence of relative error of energy norm versus the number of integration points for Figure 20.

MLS-based element increases. We see that the MLS-based variable-node element passes the patch test, such that the bending stress, which is linear in the beam height co-ordinate, is correctly captured.
3.2. A cantilever problem

A cantilever beam is selected for evaluating the accuracy of quadratic-type MLS-based elements. The cantilever beam is depicted in Figure 23. The exact displacements are given in Reference [21] as

\[ u_x = -\frac{P}{6EI} \left( y - \frac{D}{2} \right) [3x(2L - x) + (2 + v)y(y - D)] \]  

\[ u_y = \frac{P}{6EI} \left[ x^2(3L - x) + 3v(L - x) \left( y - \frac{D}{2} \right)^2 + \frac{4 + 5v}{4} D^2x \right] \]  

where \( I = D^3/12 \) and properties are \( D = 4.0 \) m, \( L = 8.0 \) m, \( L_1 = 4.0 \) m, \( E = 1 \times 10^6 \) Pa, \( v = 0.25 \). Two different mesh sizes are used for \( \Omega_2 \) as in Figure 24. The traction \( P = 1 \times 10^4 \) N is applied at the right-end of beam and the left-end of beam is fixed in the horizontal direction. The contour plots of \( \sigma_{xx} \) are shown in Figure 24 for two different mesh sizes. The convergence of energy norm in the domain, \( \Omega_2 \), is plotted in Figure 25 compared with that of the linear-type MLS-based element reported in Reference [19]. The relative error of energy norm is computed.
Figure 25. The relative error of energy norm in the domain $\Omega_2$ of a cantilever beam with non-matching interface using the quadratic-type MLS-based element, compared with the results using the linear-type MLS-based element in Reference [19].

by the domain integral over the left-half domain, $\Omega_2$. Figure 25 shows that the relative error of energy norm of the present MLS-based element compatible with quadratic interpolation is close to zero or negligibly small in comparison to those of the MLS-based variable-node element compatible with linear interpolation.

Before closing, we want to add the following. In the course of review, the reviewer pointed out that the present computational scheme may not be applicable for incompressible materials because of the large number of integration points used in this method. In principle, this problem may be resolved by decomposing deformation into two parts: deviatoric part and volumetric part. Actually, more complex locking phenomena may appear when the present method is applied for plate and shell elements, which contain the out-of-plane bending and transverse shear. The locking problem itself is a very complex issue when it comes to plate and shell structures, particularly for mesh-free methods [22], and it is not be answered straightforwardly, but it is another in-depth subject of research, which we are currently working on.

4. CONCLUDING REMARKS

In this study, new variable-node quadratic elements are developed based on the MLS approximation. The shape functions retain the quadratic interpolation along the element boundary while they involve complex rational function in the element interior. The accuracy and effectiveness of the elements is illustrated via the patch test and the cantilever beam model consisting of two non-matching meshes. The present study demonstrates the usefulness of extending the
trial function space to the rational function with the aid of MLS approximation, which makes it possible to implicitly construct the shape functions beyond the limitation of the explicit polynomial-type shape function of the regular finite elements. The diversity and generality of the proposed elements in adjusting shape functions will enable us to find various applications in many of physics and mechanics problems, particularly, in a class of problems entailing discontinuities. One profound example is the simulation of propagating cracks, which is reported in Part II of this work [20].

APPENDIX A

In mesh-free method, the shape functions are constructed by using the MLS approximation. The MLS interpolant, $u^h(x)$, of $u(x)$ is written as

$$ u^h(x) = \sum_{j=1}^{m} p_j(x) a_j(x) = p^T(x) a(x) $$  \hspace{1cm} (A1) $$

where the linear and quadratic basis $p(x)$ in the two-dimensional problem are as follows:

$$ p^T(x) = [1, x, y] (m = 3) $$  \hspace{1cm} (A2) $$

$$ p^T(x) = [1, x, y, x^2, y^2, xy] (m = 6) $$  \hspace{1cm} (A3) $$

The coefficient, $a(x)$, is calculated by minimizing the weight discrete $L_2$ norm as follows:

$$ \frac{\partial J}{\partial a} = 0 $$

$$ J = \sum_{I=1}^{NP} w(x - x_I) (p^T(x) a(x) - u_I)^2 $$  \hspace{1cm} (A4) $$

where $NP$ is the number of nodes having the non-zero value of the weight function, $w(x - x_I)$ and $u_I$ is the value of $u$ at $x = x_I$. Following equations are obtained by Equation (A4):

$$ A(x)a(x) = B(x)u $$

$$ a(x) = A^{-1}(x)B(x)u $$  \hspace{1cm} (A5) $$

where

$$ A(x) = p^T W(x) p $$

$$ B(x) = p^T W(x) $$

$$ u = [u_1, u_2, \ldots, u_{NP}] $$

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With the aid of Equation (A5), Equation (A1) is now written as

\[ \mathbf{u}^b(x) = \mathbf{p}^T(x) \mathbf{a}(x) = \mathbf{p}^T(x) \mathbf{A}^{-1}(x) \mathbf{B}(x) \mathbf{u} \]

\[ \mathbf{u}^b(x) = \sum_{I=1}^{NP} \phi_I(x) u_I \]

where

\[ \phi_I(x) = \sum_{j=1}^{m} p_j(x) (\mathbf{A}^{-1}(x) \mathbf{B}(x))_{jI} \] (A6)

Equation (A6) are the shape functions constructed by using MLS approximation [23].

In this study, we have determined the domains of weight functions in accordance with Equation (5). The goal is to construct the shape functions that are compatible with each other or conventional quadratic elements at the boundaries of elements. For simplicity, let us consider the construction of one dimensional shape functions. When the positions of nodes are like those in Figure A1, we can obtain the shape functions using MLS approximation as follows:

\[ \phi_i = \frac{N_i}{D}, \quad i = 1, 2, 3, 4, 5 \] (A7)

where

\[ N_1 = W_1[6W_4W_5 + \zeta(-3W_4 - 18W_5)W_2 + (-18W_4W_5 + (-32W_5 - 3W_4)W_3 + (W_3 - 18W_5)W_2 + (12W_4W_5 + (6W_4 + 32W_5)W_3 + (2W_3 + 12W_4 + 36W_5)W_2)\zeta)] \]

Figure A1. Five-noded one-dimensional parental domain.
\[ N_2 = W_2[3W_4W_5 + \zeta(9W_4 + 48W_5)W_1 + (-9W_4W_5 + (-W_4 - 12W_5)W_3 + (-9W_4 - 4W_5)W_1 \\
+ (6W_4W_5 + (2W_4 + 12W_5)W_2 + (-48W_4 - 18W_4 - 4W_3)W_1)\zeta)] \]

\[ N_3 = -[-W_4W_5 + \zeta(-9W_5 - W_4)W_2 + (-64W_5 - 9W_4 - W_2)W_1 + (-9W_2W_5 + 3W_4W_5 \\
+ (-3W_2 + 9W_4)W_1 + (-2W_4W_5 + (18W_5 + 4W_4)W_2 + (64W_5 - 2W_2 + 18W_4)W_1)\zeta)]W_3 \]

\[ N_4 = W_4[9W_2W_5 + \zeta(3W_2 + 48W_5)W_1 + (4W_2W_5 + (W_3 + 9W_5)W_2 + (9W_2 + 12W_5)W_1 \\
+ (-4W_3W_5 + (-18W_5 + 2W_3)W_2 + (-48W_5 + 6W_2 + 12W_3)W_1)\zeta)] \quad \text{(A8)} \]

\[ N_5 = W_5[-3W_2W_4 + \zeta(6W_2 - 18W_4)W_1 + (3W_2W_4 - W_3W_4 + (18W_4 + 18W_2 + 32W_3)W_1 \\
+ (2W_3W_4 + (12W_4 + 6W_3)W_2 + (36W_4 + 12W_2 + 32W_3)W_1)\zeta)] \quad \text{(A9)} \]

\[ D = W_3W_4W_5 + (9W_4W_5 + (9W_5 + W_4)W_3)W_2 \\
+ (36W_4W_5 + (64W_5 + 9W_4)W_3 + (9W_4 + 2W_3 + 36W_5)W_2)W_1 \]

where \( W_i = W(x - x_i) \) is a weight function associated with node \( i \).

First, for \(-1 < \zeta < 0\), let \( W_4 = W_5 = 0 \). Then the shape functions \( \phi_i, (i = 1, 2, 3, 4, 5) \) are

\[
\phi_1 = N_1/D = [W_1(W_2W_3 + 2\zeta W_2 W_3)\zeta]/[W_1 W_2 W_3] = (1 + 2\zeta)\zeta
\]

\[
\phi_2 = N_2/D = [W_2(-4W_1 W_3 - 4\zeta W_1 W_3)\zeta]/[W_1 W_2 W_3] = -4(1 + \zeta)\zeta
\]

\[
\phi_3 = N_3/D = [-(5W_1 W_2 + (-3W_1 W_2 - 2\zeta W_1 W_2)\zeta)W_3]/[W_1 W_2 W_3] = 1 + 3\zeta + 2\zeta^2
\]

\[
\phi_4 = \phi_5 = 0
\]

Next, for \(0 < \zeta < 1\), let \( W_1 = W_2 = 0 \). Then the shape functions \( \phi_i, (i = 1, 2, 3, 4, 5) \) are

\[
\phi_1 = \phi_2 = 0
\]

\[
\phi_3 = N_3/D = [(W_4W_5 + (-3W_4W_5 + 2\zeta W_4 W_5)\zeta)W_1]/[W_3 W_4 W_5] = 1 - 3\zeta + 2\zeta^2
\]

\[
\phi_4 = N_4/D = [W_4(4W_3 W_5 - 4\zeta W_3 W_5)\zeta]/[W_3 W_4 W_5] = -4(-1 + \zeta)\zeta
\]

\[
\phi_5 = N_5/D = [W_5(-W_3 W_4 + 2\zeta W_3 W_4)\zeta]/[W_3 W_4 W_5] = (-1 + 2\zeta)\zeta
\]

These shape functions are plotted in Figure A2 and the condition for the compatible shape functions is written as follows:

\[
W_1 = 0 \quad \text{where} \quad 34, 45
\]

\[
W_2 = 0 \quad \text{where} \quad 34, 45
\]

\[
W_4 = 0 \quad \text{where} \quad 12, 23
\]

\[
W_5 = 0 \quad \text{where} \quad 12, 23
\]

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where $W_i$ denotes the weight function with which node $i$ is associated, and $ij$ represents the line segment between node $i$ and node $j$. For the two-dimensional parental domain, the weight functions are given in the form of the product of the aforementioned one-dimensional weight functions, as given in Equation (2).

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