An application of two-state $M$-integral for computing the intensity of the singular near-tip field for a generic wedge

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Abstract

The two-state $M$-integral is applied for computing the intensity of the singular near-tip field around the vertex of a generic composite wedge. The eigenfunction expansion is used together with an energetics argument associated with the $M$-integral to show that a complementary eigenfield exists for every eigenfunction in a generic wedge. The proposed computational scheme is effective in finding the complete eigenfunction expansion, including the dominant singular terms along with the higher order terms as well. The present method is highly efficient and simple to use: the near-field information for the singular elastic boundary layer can be extracted from the far-field data without having to deploy singular finite elements for the wedge vertex. An exemplary case is illustrated by the re-entrant edge of a thin-film segment bonded to a substrate. The local stress intensity at the re-entrant vertex is obtained in terms of the shear stress intensity based upon the membrane model for the thin film on the substrate. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Three conservation laws for two equilibrium states, termed the two-state conservation integrals, were proposed by Chen and Shield (1977). Among these three two-state conservation laws, the two-state $J$-integral has been widely employed for obtaining stress intensities and elastic $T$-stresses for cracks, as well as for finding dislocation strength. A large number of papers have been published on this subject (e.g., Yau et al., 1980; Shih and Asaro, 1988; Kfouri, 1986; Choi et al., 1992). The two-state $L$-integral was employed by Choi and Earmme (1992) to compute stress intensities for circular arc-shaped cracks.

The two-state conservation laws, in conjunction with a straightforward displacement-based FEM, provide an efficient tool for calculating stress intensities and elastic $T$-stresses for cracks, or for calculating dislocation strength. This method is capable of extracting the near-tip information directly from the far-field deformation. This is a major advantage over the singular finite elements (see Hughes, 1987 or references cited therein), and other various special techniques such as the boundary collocation method (Wang, 1984), the singular hybrid FEM (Tong et al., 1973), or the enriched FEM (Jeon et al., 1996). Similar remarks can be made for computing elastic $T$-stress. In contrast to such applications of the two-state $J$- and $L$-integrals, however, applications of the two-state conservation law from the $M$-integral have not been discussed in the pertinent literature.

The purpose of the present work is to examine the two-state $M$-integral in relation to a scheme for computing the intensities of singularities for a generic isotropic wedge. This particular geometry contains a variety of elements: free edges, cracks terminating at a material interface, and re-entrant edges of a thin film. It is found that an energetics property of the $M$-integral places a restriction upon the structure of asymptotic solutions in the eigenfunction series for these generic wedges in which the $M$-integral becomes path independent. It turns out that this makes the present generic wedge problem easily tractable by use of the two-state $M$-integral.

We first briefly review the two-state conservation laws from the $J$-integral and the $M$-integral (Chen and Shield, 1977). Subsequently we discuss how the two-state conservation law from the $M$-integral is applied for finding intensities of singularities for the singular elastic boundary layers in the aforementioned class of wedges. The success of this scheme is crucially linked to the existence of the auxiliary solutions in the form of the complementary eigenfunction in the $M$-integral sense. The existence of these complementary solutions is then proved, with the aid of the eigenfunction series, for the generic wedge under consideration. The weight functions for the special cases of the present generic wedges were obtained in Sham and Bueckner (1988) and Wu and Chang (1993). The proof in this paper elucidates that the weight function for every eigenfunction has the same form as the eigenfunction with a different eigenvalue. This means that the auxiliary field, used by Sinclair et al. (1984) in applying Betti’s reciprocal theorem for computing stress intensity at sharp notches, can be obtained in the form of this complementary eigenfunction. In conjunction with a straightforward
displacement-based FEM, this application is demonstrated for a re-entrant edge of a thin film. The present scheme is straightforward and simple, and this methodology gives an efficient and robust tool for solving the elastic boundary layer problems of the generic composite wedge with singularities.

2. Two-state conservation laws

In this section, we discuss the two-state conservation laws introduced by Chen and Shield (1977). For clarity, we restrict our attention to plane strain problems, and assume that the material under consideration is isotropic.

Let $u_a$, $\sigma_{ij}$ and $\varepsilon_{ij}$ denote Cartesian components of a displacement vector, a stress and strain tensor, respectively. For the plane strain problems, we have the following governing equations in a two-dimensional domain:

\[
\sigma_{\alpha\beta} = 0 \quad (\alpha, \beta = 1, 2),
\]

\[
\varepsilon_{\alpha\beta} = (u_\alpha, \beta + u_\beta, \alpha)/2
\]

\[
\sigma_{ij} = C_{ijkm}\varepsilon_{km}, \quad C_{ijkm} = \mu\delta_{ik}\delta_{jm} + \mu\delta_{im}\delta_{jk} + 2\nu\mu\delta_{ij}\delta_{km}/(1 - 2\nu),
\]

where $\mu$ and $\nu$ are shear modulus and Poisson’s ratio, respectively, and the comma indicates the partial differentiation with respect to the Cartesian coordinate $x_i$. For the plane problems, the $J$-integral (Eshelby, 1956; Rice, 1968) and the $M$-integral (Knowles and Sternberg, 1978) may be written as:

\[
J = \int_C \left( Wn_j - t_j \frac{\partial u_i}{\partial x_j} \right) ds,
\]

\[
M = \int_C \left( Wn_j - t_j \frac{\partial u_i}{\partial x_j} \right) x_j ds,
\]

where $n_j$ is the component of unit outward normal on the contour $C$; $W$ and $t_j$ indicate the strain energy density and the traction component, given as $W = C_{ijkm}\varepsilon_{km}/2$ and $t_j = \sigma_{ij}n_j$. In our discussion, we exclude the $L$-integral and the associated two-state integral because they are not connected to the present work in a pertinent way.

Consider two independent elastic states ‘A’ and ‘B’ for the plane problems. We obtain another elastic state ‘C’, by superposing the two equilibrium states ‘A’ and ‘B’. Then the path-independent integrals $J$ and $M$ are written as

\[
J^C = J^A + J^B + J^{(A,B)},
\]

\[
M^C = M^A + M^B + M^{(A,B)},
\]
where the superscripts $A$, $B$ and $C$ indicate the aforementioned elastic states, and $J^{(A,B)}$ and $M^{(A,B)}$ are given as

$$J^{(A,B)} = \int_C \left[ C_{ijkl} C_{ij} C_{kl} n_1 \right] \left( \frac{\partial u_i^B}{\partial x_1} + t_i^B \frac{\partial u_j^A}{\partial x_1} \right) \, ds,$$

(4a)

$$M^{(A,B)} = \int_C \left[ C_{ijkl} C_{ij} C_{km} n_1 \right] \left( \frac{\partial u_i^A}{\partial x_1} + t_i^A \frac{\partial u_j^A}{\partial x_1} \right) \, ds.$$

(4b)

The integrals $J^{(A,B)}$ and $M^{(A,B)}$ result from the mutual interaction between two elastic states $A$ and $B$. These are conservation integrals for two equilibrium states, since the area integral version of these contour integrals vanishes identically for the domains with no singularities. In this context, these may be termed the two-state conservation laws, as in the preceding discussion.

In contrast to the extensive applications of the two-state conservation integral $J^{(A,B)}$ for crack problems, the applications of the integral $M^{(A,B)}$ have not been discussed in the literature. In fact, the broad applicability of $M^{(A,B)}$ will assert itself in our subsequent development. The following section presents the application of $M^{(A,B)}$ to the problems of a generic isotropic wedge, and these are inclusive enough to cover another class of problems for which the approach based upon $J^{(A,B)}$ is not applicable.

3. Application of the two-state conservation integral $M^{(A,B)}$ to a generic isotropic wedge

In this section, we describe a generic isotropic composite wedge, and examine the application of the two-state conservation integral $M^{(A,B)}$ for finding the intensities of stress singularities at the singular elastic boundary layer of the wedge. There have been several works in relation to calculating the notch or wedge intensity factors. For example, Sham and Bueckner (1988) applied the notion of the weight function to an isotropic-bimaterial notch under antiplane loading. Essentially along the same line of approach via Betti’s reciprocal theorem, Wu and Chang (1993) obtained the intensity of a near-tip singular stress field due to a concentrated line load and dislocation for a notched elastic body composed of a single homogeneous material. A problem of an elastic wedge composed of a single homogeneous material was treated, with the aid of Betti’s reciprocal theorem, by Sinclair et al. (1984). Leguillon and Sanchez-Palencia (1987) examined a wedge problem in the framework of singularity theory for elliptic boundary value problems. They introduced a bilinear functional and established a variational principle appropriate for elastic fields containing singularities. In this section, we are concerned with the application of the two-state $M$-integral for examining a stress field near the vertex of a generic composite wedge. We restrict ourselves to an isotropic composite wedge. The extension of
the present discussion to the case of an anisotropic composite wedge, or to the case of antiplane deformation can be made along the same lines.

Consider a generic composite wedge composed of \( N \) linear elastic materials which are isotropic, as shown in Fig. 1. Such a generic \( N \)-material composite wedge was first considered by Dempsey and Sinclair (1979) in relation to finding stress singularities at the wedge vertex. Let \( \mu^{(n)}, \nu^{(n)} \) and \( \phi^{(n)} \) \( (n = 1 \sim N) \) denote the elastic properties and the interface angle of each sector, respectively. Note that one can form a notch or a wedge with traction-free faces by choosing zero stiffness on one of the sectors. Some specific choices of \( N, \mu^{(n)} \) and \( \phi^{(n)} \) lead to well-known problems, such as, interfacial cracks, wedges or notches with traction-free faces, free edges in composite layers, or re-entrant edges of thin films and cracks arrested at a material interface. The situations in which these particular geometries are applicable may be found in the mechanics of thin films and composite materials.

Fig. 1. A generic composite wedge composed of \( N \) isotropic elastic materials.
electronic packaging. Note that the conservation integral $J$ and the associated two-state conservation integral $J^{(A,B)}$ is not path-independent for the present generic composite wedges except for the case of (interfacial) cracks. Accordingly, it is impossible to apply the approach via $J^{(A,B)}$ for finding the intensities of singularities for generic composite wedges other than (interfacial) cracks.

We now turn our attention to the $M$-integral in Eq. (2b) and its associated two-state integral $M^{(A,B)}$ in Eq. (4b) for the preceding generic composite wedge. Recalling that the $M$-integral is dependent upon the origin of the Cartesian coordinate system $(x_1, x_2)$, we take its origin at the wedge vertex, as shown in Fig. 2. Furthermore, we will employ polar coordinates $(r, \theta)$ whenever it is convenient to do so.

Consider two circular contours $C_1$ and $C_{II}$ as shown in Fig. 2, where we take $N = 5$ for the sake of definiteness, but the conclusion to follow is valid for the case of an arbitrary $N$. We assume that all neighboring sectors are rigidly joined along the material interface, so that the traction and the displacement are

![Fig. 2. Two contours $C_1$ and $C_{II}$ for the generic composite wedge with $N = 5$.](image)
continuous across the interfacial boundary, which is a radial line from the origin as shown in Fig. 2. From the continuity of traction and displacement, we can show that the $M$-integral of Eq. (2b) is path independent for the composite wedge of Fig. 2, i.e.,

$$M(C_{II}) = M(C_I).$$

The path-independence of the two-state $M$-integral $M^{(A,B)}$ for the present wedge problem is apparent from the above and Eq. (3b). For wedges with traction-free faces, the contributions from the traction-free faces to the $M$ and $M^{(A,B)}$ integrals vanish identically, and the closed contour $C_I$ and $C_{II}$ will be opened contours. In passing, we remark that the aforementioned path-independence of the $M$-integral for a homogeneous wedge with a concentrated load or with a dislocation at its vertex was used by Freund (1978) for computing the stress intensity factors for cracks of special geometries and loadings in infinite planar domains.

Before we apply the two-state $M$-integral for obtaining the intensities of singularities for this composite wedge, we briefly summarize the structure of the asymptotic solutions in the form of eigenfunction series for the wedge problem. The asymptotic solutions for the stress and displacement components for the present wedge may be written in the following power type eigenfunction of $z = x + iy$ with $\bar{z} = x - iy$ (see Appendix A for derivation):

$$\sigma_{x\beta}^{(m)} = \text{Re} \left[ \sum_{\sigma} \beta_n \sum_{k=1}^{2} C_{kn}^{(m)} (A_{jk} g_{n}(\bar{z})) + \Gamma_{jk} \bar{z} g_{n}(\bar{z})) + C_{kn}^{(m)} (A_{jk} g_{n}(\bar{z}) + \bar{A}_{jk} \bar{z} g_{n}(\bar{z})) \right],$$

$$u_\alpha^{(m)} = \frac{1}{2\mu^{(m)}} \text{Re} \left[ \sum_{\sigma} \beta_n \sum_{k=1}^{2} C_{kn}^{(m)} (p_{jk} g_{n}(\bar{z}) + q_{jk} \bar{z} g_{n}(\bar{z})) \right],$$

with $g_{n}(\bar{z}) = z^n$ and the non-zero components of $A_{jk}, \Gamma_{jk}, p_{jk}$ and $q_{jk}$:

$-A_{111} = A_{221} = iA_{121} = 1, \quad A_{112} = A_{222} = 2,$

$\Gamma_{112} = -\Gamma_{222} = -i\Gamma_{122} = -1,$

$p_{11}^{(m)} = -ip_{21}^{(m)} = -1, \quad p_{12}^{(m)} = ip_{22}^{(m)} = 3 - 4\nu^{(m)}, \quad q_{12} = -iq_{22} = -1,$

(5)
where $\delta_n$ is an eigenvalue and $C_{kn}$, short for $C_k(\delta_n)$, is the corresponding eigenvector; $\beta_n = \beta(\delta_n)$ represents the load parameter or the intensity of the elastic field associated with eigenvalue $\delta_n$. Note that $\beta_n$ is real for a real $\delta_n$, but it is, in general, complex for a complex $\delta_n$. For a complex $\delta_n$, it is self-evident from the expression (5) that its conjugate $\delta_n$ also belongs to the eigenvalues. For clarity, we assume that the imaginary part of complex $\delta_n$ is positive in the expression (5) since a complex eigenvalue $\delta_n$ and its conjugate $\delta_n$ lead to the same eigenfunction.

The superscript ‘(m)’ indicates the $m$-th sector. We will omit the superscript ‘(m)’ for simplicity unless it is needed for clarity to distinguish one sector from another.

For the generic composite wedge under consideration, the boundedness of the strain energy restricts the range of eigenvalue $\delta_n$ such that $\text{Re}(\delta_n) > -\frac{1}{2}$. The terms giving rise to a singular stress field near the vertex are represented by the eigenvalues $\delta_s$ within the range $-1 < \text{Re}(\delta_s) < 0$.

With the aid of the foregoing asymptotic solution (5), we now proceed to apply the two-state $M$-integral to the generic wedge problem. For an arbitrary eigenvalue $\delta_l$ in Eq. (5), we first define its complementary eigenvalue $\delta_c^l$ in the $M$-integral sense such that

$$\delta_c^l + \delta_l = -2.$$

It is not clear yet whether or not $\delta_c^l$ is another eigenvalue for the eigenvalue problem in the given generic wedge. As will be shown later, however, it turns out that $\delta_c^l$ constitutes another eigenvalue as long as $\delta_l$ belongs to the eigenvalues for the given wedge problem. We just proceed by assuming this for the time being, but this will be proved later, and will be verified numerically with some examples.

The elastic state for the complementary eigenvalue $\delta_c^l$, with the load parameter $\beta_c^l$, may be written as

$$\sigma_{\text{eff}}(\delta_c^l) = \text{Re} \left[ \beta_c^l \sum_{k=1}^{2} \left( C_{kl}(A_{zjk}g_{l}'(z) + \Gamma_{zjk}\bar{z}g_{l}''(z)) + C_{(k+2)l}(\bar{A}_{zjk}g_{l}'(\bar{z}) + \bar{\Gamma}_{zjk}\bar{z}g_{l}''(\bar{z})) \right) \right],$$

$$u_{z}(\delta_c^l) = \frac{1}{2\mu} \text{Re} \left[ \beta_c^l \sum_{k=1}^{2} \left( C_{kl}(p_{zjk}g_{l}(z) + q_{zjk}\bar{z}g_{l}''(z)) + C_{(k+2)l}(\bar{p}_{zjk}g_{l}(\bar{z}) + \bar{q}_{zjk}\bar{z}g_{l}''(\bar{z})) \right) \right].$$
with \( l \) being the index indicating the eigenvalue \( \delta_j \), so that \( g_{l}(z) \)

\[
= z^{\delta_j} \quad \text{and} \quad C_{kl} = C(\delta_j).
\]  

(6)

Our motivation for introducing such an auxiliary field will be apparent from the subsequent development in the application of the two-state integral \( M^{(A,B)} \).

We now consider the superposition of the above complementary elastic state onto the given elastic state for the generic wedge under consideration. In Eq. (4b), we substitute the elastic field \( (5) \) of the generic wedge for the elastic state \( (A) \), and the complementary field \( (6) \) for the elastic state \( (B) \). For the line integral path, we choose a circular arc or contour with a radius \( \hat{r} \). We arrange the eigenvalues \( \delta_n \) such that \( \Re(\delta_1) < \Re(\delta_2) < \Re(\delta_3) < \cdots \). We then are led to the following expression for the two-state integral \( M^{(A,B)} \) after some algebra:

\[
M^{(A,B)} = \sum_{n=1}^{\infty} \int_C \Re[\hat{p}^{\delta_n + \hat{\delta}_j + 2} \beta_n \beta_j F(\delta_n, \delta_j^c, 0) + \hat{p}^{\delta_n + \hat{\delta}_j + 2} \beta_n \beta_j F(\delta_n, \delta_j^c, 0)] \, d\theta,
\]

where \( F(\delta_n, \delta_j^c, \theta) \) is defined in each sector and derived in Appendix B. We exploit the path-independence of \( M^{(A,B)} \) to show that the contributions of all \( \delta_n \) to the integral vanish identically except for \( \delta_n = -2 + \delta_j^c = \delta_j \) i.e., we take \( \hat{r} \) to go to infinity for \( \Re(\delta_n + \delta_j^c) = \Re(\delta_n + \delta_j) < -2 \) while we take \( \hat{r} \) to be arbitrarily small for the case of \( \Re(\delta_n + \delta_j^c) \geq -2 \). Let \( \beta_j^c = \exp(i \Psi) \) and \( \beta_j = a_j - i b_j \) where \( \Psi, a_j \) and \( b_j \) are real. Then, the above relation finally reduces to (see Appendix B for detail)

\[
M^{(A,B)} = a_j I_a(\Psi, \delta_j) + b_j I_b(\Psi, \delta_j)
\]  

(7)

with

\[
I_a(\Psi, \delta_j) = \int_{\phi_1}^{\phi_2} \Re[e^{i \Psi} F(\delta_j, \delta_j^c, \theta) + e^{-i \Psi} F(\delta_j, \delta_j^c, \theta)] \, d\theta,
\]

\[
I_b(\Psi, \delta_j) = \int_{\phi_1}^{\phi_2} \Im[e^{i \Psi} F(\delta_j, \delta_j^c, \theta) + e^{-i \Psi} F(\delta_j, \delta_j^c, \theta)] \, d\theta,
\]

where \( \phi_1 \) and \( \phi_2 \) indicate the angular boundary of the domain for the generic wedge. The integrals \( I_a(\Psi, \delta_j) \) and \( I_b(\Psi, \delta_j) \) are readily computed by employing Simpson’s rule for a prescribed \( \Psi \). We prescribe two different values of \( \Psi \), most conveniently \( \Psi = 0 \) and \( \Psi = \pi/2 \), to compute the corresponding two sets of values of \( I_a \) and \( I_b \). Then Eq. (7) provides a set of two linear equations that determine a complex load parameter \( \beta_j = a_j - i b_j \) when \( M^{(A,B)} \) on the left-hand side of Eq. (7), corresponding to each of the two chosen values of \( \Psi \), is evaluated from the far-field. Note that we have only to take \( \Psi = 0 \) for a real \( \delta_j \) since \( \beta_j \) is real. The two-state integral \( M^{(A,B)} \) is accurately calculated on the far field via a displacement-
based FEM in conjunction with the following domain integral representation (Li et al., 1985; Nikishkov and Atluri, 1987):

$$M^{(A, B)} = \int_{A_{II} - A_I} \left[ C_{ijkm} A^B_{jkm} x_j - \sigma^B_{ij} \frac{\partial u^B_j}{\partial x_i} x_i - \sigma^B_{ij} \frac{\partial u^B_i}{\partial x_j} x_j \right] q_{ij} \, dA,$$

where $A_I$ and $A_{II}$ represent the domain inside each path of $C_I$ and $C_{II}$, respectively, and accordingly $A_{II} - A_I$ indicates the annular region surrounded by the paths $C_I$ and $C_{II}$ (see Fig. 2); the function $q(x_1, x_2)$ is a weight defined such that it is 1 on $C_I$, and it linearly decreases to 0 on $C_{II}$.

In the foregoing development, after taking $d_{cl} = d_{cs} = 0$ in Eq. (6), we can then calculate the intensity $\beta_n$ for the stress singularity $\delta_l$ at the vertex of the generic composite wedge. Apparently this is the case for all higher order load parameters $\beta_n$ associated with eigenvalues $\delta_n$.

The two keys for success of this approach are the path-independence of the $M^{(A, B)}$ for the generic wedge problem, and the existence of an auxiliary elastic state in the form of the complementary field (6). We have shown that the two-state integral $M^{(A, B)}$ is path-independent, but we assumed from the outset the existence of the complementary elastic state (6) for every eigenvalue of $\delta_l$. It is important to note that the elastic field (6) already satisfies the equilibrium equation, the compatibility condition, and the stress–strain law. Accordingly to prove the existence of the complementary solution (6), we only need to show that it satisfies the near-field conditions given by Eq. (A1) in Appendix A. That is, we shall show that the complementary eigenvalue $\delta_{cl}^l$ is an eigenvalue for the eigenvalue problem resulting from the near-field condition (A1) in the generic wedge whenever $\delta_l$ is an eigenvalue for that same problem. To this end, we utilize an energetics argument as follows. We first consider the eigenfunction associated with one arbitrary eigenvalue $\delta_l$, the real part of which is restricted to be greater than $-1$ due to the boundedness of the strain energy, for a generic wedge. Firstly, suppose the generic wedge is in an elastic state resulting from this single eigenfunction, wherein the stress and displacement fields may be written as:

$$\sigma_{xj}(\delta_l) = \text{Re} \left[ \beta^l \sum_{k=1}^{2} \left[ C_{kl}(A_{xjk} e^{i\delta_l e^{i\theta}} + \Gamma_{xjk} e^{i(\delta_l - 2\theta)})
+ C_{(k+2)l}(A_{xjk} e^{-i\delta_l e^{i\theta}} + \Gamma_{xjk} e^{-i(\delta_l - 2\theta)}) \right] \right],$$
\[ u_x(\delta_j) = \text{Re} \left[ \frac{\beta_j}{2\mu} \rho_{\delta_j+1} \sum_{k=1}^{2} \left\{ C_{k} \left( p_{2k} e^{i(\delta_j+1)\theta} + q_{2k} e^{i(\delta_j-1)\theta} \right) \right. \right. \\
+ C_{(k+2)} \left( p'_{2k} e^{-i(\delta_j+1)\theta} + q'_{2k} e^{-i(\delta_j-1)\theta} \right) \left\} \right]. \] (9)

The elastic state of this wedge, which we hereafter refer to as the first generic wedge, may be produced by prescribing the far-field displacement corresponding to Eq. (9). Note that the above elastic state is self-equilibrated, and the resultant traction around an arbitrary closed contour vanishes identically. Secondly, we will consider another generic composite wedge, which has the same geometry as the first wedge with the exception that it has a tiny cylindrical cavity or a circular keyhole free from traction and centered at the wedge vertex. Furthermore, let this wedge be under the same displacement boundary condition in the far field as the

Fig. 3. The generic composite wedge with a tiny circular cylindrical hole of a radius \( r^* \) at the vertex \( (C^* \text{ indicates the circular boundary of the cylindrical cavity}). \)
first wedge. Hereafter we will call this the second generic wedge. This second generic composite wedge may be thought of as follows: we first consider the same composite wedge as the first generic wedge in loading as well as in geometry. While freezing the displacements on the far-field boundary, we now create a circular cylindrical cavity with a very small but nonvanishing radius $r^*$, free from traction and centered at the wedge vertex (see Fig. 3). Since $r^*$ is very small, the asymptotic solution for this wedge will still be determined from the same conditions (A1) as for the first wedge. To meet the traction-free condition on the cavity surface $r = r^*$, however, the displacement field for the second wedge will be given as a superposition of the elastic state of Eq. (9) and another elastic state whose displacement field is given by the eigenfunction expansion of a Laurent series type, obtained by taking $-\infty < \text{Re}[\delta_n] < -1$ in Eq. (5). This assumes an essential similarity to the case of a crack under small-scale yielding: in the situation of small-scale yielding an eigenfunction expansion of negative powers occurs in addition to the inverse square singularity, which represents the leading term in the outer solution (see Hui and Ruina, 1995). Note that the appearance of any eigenvalue $\text{Re}[\delta_n] \geq -1$, except for $\delta_1$, in the expansion for the second generic wedge, would yield a finite displacement at the boundary and therefore would violate the displacement boundary condition on the far-field, where the displacement field is frozen according to Eq. (9). The elastic field for the second generic wedge may then be written as:

$$
\sigma_{z\theta} = \sigma_{z\theta}(\delta) + \text{Re} \left[ \sum_{\text{Re}[\delta_n] < -1}^{\infty} \beta_n \beta_n^{p_k} \sum_{k=1}^{2} \left\{ C_{kn} \left( A_{z\theta k} e^{i\delta_k\theta} + \Gamma_{z\theta k} e^{i(\delta_k-2)\theta} \right) \right. 
+ C_{(k+2)\theta} \left( A_{z\theta k} e^{-i\delta_k\theta} + \Gamma_{z\theta k} e^{-i(\delta_k-2)\theta} \right) \right\} \right],
$$

$$
u_{\theta z} = u_{\theta z}(\delta) + \frac{1}{2\mu} \text{Re} \left[ \sum_{\text{Re}[\delta_n] < -1}^{\infty} \beta_n \beta_n^{p_k+1} \sum_{k=1}^{2} \left\{ C_{kn} \left( p_{nk} e^{i(\delta_n+1)\theta} \right) 
+ q_{nk} e^{i(\delta_n-1)\theta} \right\} + C_{(k+2)\theta} \left( p_{nk} e^{-i(\delta_n+1)\theta} + q_{nk} e^{-i(\delta_n-1)\theta} \right) \right\}. \tag{10}
$$

where $\sigma_{z\theta}(\delta)$ and $u_{\theta z}(\delta)$ are given by Eq. (9), and the second terms on the right-hand sides represent the eigenfunction expansion due to the cavity. Now we consider the potential energy trend of the second wedge with respect to the size of the cavity $r^*$. Let $\pi[r_i^*]$ denote the potential energy for the wedge with the cavity radius $r_i^*$ ($i = 1, 2$). We suppose $0 < r_1^* < r_2^*$, and then let $\pi[0]$ indicate the potential energy for the first generic wedge, which has no cavities. Utilizing the principle of a minimum potential energy, we can show $\pi[0] > \pi[r_1^*] > \pi[r_2^*]$ for an
arbitrary \( r_1^* \) and \( r_2^* \) satisfying \( 0 < r_1^* < r_2^* \). That is, we have

\[
\frac{\partial \pi[r^*]}{\partial r^*} < 0 \text{ for an arbitrary } r^* > 0.
\] (11)

This is consistent with the observation that the potential energy is a monotonically decreasing function of the cavity size \( r^* \) under a fixed boundary condition. We can now turn our attention to the physical interpretation that the \( M \)-integral is the rate of the potential energy release with respect to the unit expansion of a two-dimensional cavity, free from traction (Budiansky and Rice, 1973). Then we can show that the \( M \)-integral for the second generic wedge with a cavity radius of \( r^* \) is given by the following form:

\[
M = -\frac{\partial \pi[r^*]}{\partial r^*} r^*.
\] (12)

From this and Eq. (11), we now conclude that \( M > 0 \) for any \( r^* > 0 \) in the second generic wedge, i.e., the \( M \)-integral for the generic composite wedge with a tiny cavity, the elastic state of which is given by Eq. (10), is nonzero. Substituting the expressions (10) for the \( M \)-integral, we see that the \( M \)-integral for this wedge may be contemplated as the summation of two-state \( M \)-integrals as follows:

\[
M = \sum_{\text{Re}(\delta_j) < -1/2} \left\{ \sum_{\text{Re}([\delta_l]) \leq \text{Re}(\delta_j)} M^{(\delta_j, \delta_l)} \right\} = M^{(\delta_j, \delta^*_{\text{c}})},
\] (13)

where the eigenvalues \( \delta_j \) and \( \delta_k \) include, in addition to the eigenvalue \( \delta_k \) all the eigenvalues whose real parts are smaller than \(-1\), and

\[
M^{(\delta_j, \delta^*_{\text{c}})} = \int_{\phi_1}^{\phi_2} \text{Re} \left[ r^{\delta_j + \delta^*_{\text{c}} + 2 \beta_j \beta_k F(\delta_j, \delta^*_{\text{c}}, \theta) + r^{\delta_j + \delta^*_{\text{c}} + 2 \beta_j \beta_k F(\delta^*_{\text{c}}, \delta^*_{\text{c}}, \theta)} \right] d\theta,
\]

where \( F(\delta_j, \delta^*_{\text{c}}, \theta) \) is again given in Appendix B. The second equality in Eq. (13) stems from the same reasoning as before: by analytic continuation we first extend the elastic field from each pair of \( \delta_j \) and \( \delta_k \), either into the cavity region or into the infinite region, and then we can take an arbitrarily small circular contour or an infinite circular contour to show that all contributions vanish except for the case \( \delta_j + \delta_k = -2 \). That is, a nonzero contribution occurs only when \( \delta_k \) is the complementary eigenvalue to \( \delta_j \) in the \( M \)-integral sense, and this is possible only when \( \delta_j = \delta^*_{\text{c}} \) and \( \delta_k = \delta^*_{\text{c}} \) because all \( \delta \) except for \( \delta \in \text{range} \{ \text{Re}(\delta) < -1/2 \} \) are in the range \( \text{Re}(\delta) < -1 \). Therefore it is an eigenvalue for the eigenvalue problem resulting from the near-field conditions for the present generic wedge. This is numerically verified for
some of the practically important cases shown in Fig. 4. Table 1 shows the list of the eigenvalues for: the free edge of a layered strip composed of two joining materials (Fig. 4(a)), a re-entrant edge of a thin film bonded to a substrate (Fig. 4(b)), and a crack of which the tip is perpendicular to the interface of two-bonded materials ($\phi = \pi/2$ in Fig. 4(c)). The material properties employed for computation are given in terms of Dundurs parameters (Dundurs, 1969). We see that all the eigenvalues appear as complementary pairs in the $M$-integral sense for these examples.

Following a procedure similar to Wu (1989), we can show that the present scheme may be interpreted via the weight function approach. The notion of this weight function was applied in dealing with an isotropic bimaterial notch by Sham and Bueckner (1988). Furthermore, the present scheme is virtually the same as the approach based upon Betti’s reciprocal theorem, used by Sinclair et al. (1984), wherein the complementary field in the $M$-integral sense was chosen without the general proof from our present context. The preceding proof elucidates that the weight function and the complementary field for a generic composite wedge exist for every eigenfunction.

(a) Free edge

$$N = 3, \phi^{(1)} = \pi/2, \phi^{(2)} = 3\pi/2, \phi^{(3)} = 2\pi, \mu^{(2)} = 0$$

(b) A re-entrant edge of a thin film

$$N = 3, \phi^{(1)} = \phi, \phi^{(2)} = \pi, \phi^{(3)} = 2\pi, \mu^{(2)} = 0$$

(c) A crack arrested at a bi-material interface

$$N = 4, \phi^{(1)} = \phi^-, \phi^{(2)} = \phi^+, \phi^{(3)} = \pi, \phi^{(4)} = 2\pi, \mu^{(2)} = 0, \mu^{(1)} = \mu^{(3)}$$

Fig. 4. Specific choices of $N$, $\mu^{(n)}$, and $\phi^{(n)}$ leading to important practical cases for the generic composite wedges. (a) Free edge. $N = 3, \phi^{(1)} = \pi/2, \phi^{(2)} = 3\pi/2, \phi^{(3)} = 2\pi, \mu^{(2)} = 0$. (b) A re-entrant edge of a thin film. $N = 3, \phi^{(1)} = \phi, \phi^{(2)} = \pi, \phi^{(3)} = 2\pi, \mu^{(2)} = 0$. (c) A crack arrested at a bi-material interface. $N = 4, \phi^{(1)} = \phi^-, \phi^{(2)} = \phi^+, \phi^{(3)} = \pi, \phi^{(4)} = 2\pi, \mu^{(2)} = 0, \mu^{(1)} = \mu^{(3)}$. 
Wu and Chang (1993) solved a notch problem, wherein the intensity of a near-tip singular field in a notched planar body due to a concentrated force or a dislocation, is calculated via the approach based upon the reciprocal theorem. When this problem is approached from the viewpoint of the methodology discussed in this paper, it might provide an excellent ‘educational’ example of the application of the two-state $M$-integral.

4. The re-entrant edges of thin films which adhered to substrates

In this section, we take a re-entrant edge of a thin film segment rigidly bonded to a substrate for illustrating the application of the aforementioned scheme to an important case for the composite wedges. Consider a thin film segment rigidly bonded to a substrate, the dimension of which is sufficiently large, as compared with the film’s thickness, to be modeled as an elastic half-space (see Fig. 5). The rectangular coordinate system is taken with origin at the vertex of the left re-entrant edge. We assume that the substrate is under plane strain extension. In addition, the reflection symmetry in geometry and loading is assumed to be about $x_1=L$, so that it may be sufficient to consider only its left half. However, we emphasize that the final results of our analysis will be nondimensionalized, such that they become valid regardless of loading (except for the sign of loading) for a given vertex angle.

There have been many works with regard to the stress field for a thin film segment bonded to a substrate. It is well-known that the shear stress field on the substrate has the inverse square root singularity near the ends of a film-segment,

### Table 1

<table>
<thead>
<tr>
<th>Free edge (Fig. 4(a))</th>
<th>Re-entrant edge of a thin-film (Fig. 4(b))</th>
<th>Crack arrested at a bi-material interface (Fig. 4(c))</th>
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<tr>
<td>$3.84005 \pm i1.21597$</td>
<td>$2.50822 \pm i0.132909$</td>
<td>$2.0$</td>
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<tr>
<td>$2.82717 \pm i1.02431$</td>
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<td>$1.79578 \pm i0.865335$</td>
</tr>
<tr>
<td>$1.79768 \pm i0.866220$</td>
<td>$0.686034 \pm i0.358544$</td>
<td>$1.0$</td>
</tr>
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<td>$0.794693 \pm i0.506483$</td>
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</tr>
<tr>
<td>$-5.84005 \pm i1.21597$</td>
<td>$-4.50822 \pm i0.132909$</td>
<td>$-4.0$</td>
</tr>
</tbody>
</table>
according to the membrane or the beam model (see Freund and Hu, 1988; Erdogan and Joseph, 1990; Shield and Kim, 1992). Such a membrane or a beam model is useful for examining the stress field over the region near the end of the film-segment. However, the region concerned should be at a distance of thickness orders away from the vertex of the re-entrant edge. Furthermore, such a model does not provide any insight into the stress field in the region very close to the vertex, wherein the outer expansion due to the membrane or the beam model is not any longer valid, but an inner expansion based upon a two-dimensional continuum model is required. We examine the inner expansion of the local singular stress field near the vertex of the re-entrant edge with the aid of the two-state $M$-integral described in Section 3.

The shear stress intensity factor $k$, based upon the membrane model and giving the interfacial shear stress near the left-end of a film-segment of the relation

![Diagram of a thin-film segment bonded to a substrate under a uniform plane strain extension along the $x_1$ axis.](image-url)
\[ \sigma_{12} = k/\sqrt{L + x}, \]

is related to the \( J \)-integral in the following way (Shield and Kim, 1992):

\[ J = \frac{\pi(1 - v_2^2)}{2(1 + v_2^2)\mu_2^2} k^2, \]

where \( \mu_2 = \mu^{(2)} \) and \( v_2 = v^{(2)} \) denote the shear moduli and Poisson’s ratio of the lower-layer or the substrate. The \( k \)-field due to this membrane model constitutes the outer expansion, with its zone at a distance greater than the film-thickness dimension from the vertex of the re-entrant edge. As the vertex of the re-entrant edge is approached, however, the stress field will begin to deviate from the \( k \)-field, and it will be given by the asymptotic wedge solution (5). That is, the expression (5) provides the inner expansion near the vertex of the re-entrant edge. Note that the stress intensity \( k \) from the outer expansion based upon the membrane model will also play the role of a loading parameter for the inner solution (5).

For the re-entrant edge of a thin film segment in general, the stress singularity \( \delta_s \) with \( -1 < \text{Re}[\delta_s] < 0 \) appears as real, as shown in Table 1. Therefore the stress singularity \( \delta_s \) is taken to be real hereafter and so is the associated load parameter \( \beta_s \). We now introduce the re-entrant edge intensity \( K \) such that

\[ \sigma_{22} + i\sigma_{12} = K e^{\phi\beta_s} \text{ with } K = |K| \text{ as } r \to 0 \text{ along the interface}, \]

where \( K \) is related to the load parameter \( \beta_s \) and the corresponding eigenvector \( C_{a\delta_s} = C_{a}(\delta_s) \) for a singular stress field in the following way:

\[ K = 2 |\beta_s| [\text{Re}(-iC_{1\delta_s} - iC_{2\delta_s})]^2 + (\text{Re}(C_{1\delta_s} + 2C_{2\delta_s} + \delta_sC_{2\delta_s}))^2]^{1/2}. \]

Furthermore, \( \omega \) represents the mode mixity of the interfacial traction and we let it lie between \( -\pi \) and \( \pi \).

We relate the intensity \( K \) to the loading parameter \( k \) by introducing a dimensionless function \( K^* \) as follows:

\[ K^* = \frac{K\sqrt{\rho h}}{k} = \sqrt{\frac{\pi(1 - v_2^2)h}{2\mu_2^2 J}} h^\rho K. \]

Note that the stress singularity \( \delta_s \), dimensionless function \( K^* \), and the mode mixity \( \omega \) depend only upon the Dundurs parameters \( \alpha \) and \( \beta \) for a given \( \phi \), other than that \( \omega \) is changed to \( \omega + \pi \) or \( \omega - \pi \) when the direction of loading is reversed. The implicit function yielding the re-entrant edge stress intensity \( K \) in terms of \( \alpha, \beta, \) and \( \phi \) will be universal, regardless of the types of loading and far-field geometry, as far as the thickness of a thin-film is sufficiently small compared with the film length and the substrate dimension. For a given loading parameter \( k \) on the far field, the function \( K^* \) provides information upon \( K = |K| \), which is a scaling parameter governing the severity of the stress field near the vertex.

We compute the nondimensionalized function \( K^* \) with the aid of the two-state
\(M\)-integral together with a displacement-based FEM, as described in Section 3. The geometry and loading shown in Fig. 5 is chosen, with varying elastic properties and vertex angles, for computing \(K^*\) vs \(\alpha\) and \(\beta\), and the vertex angle \(\phi\). The results are tabulated in Table 2. As seen in Table 2, there exist two singularities for \(\phi = \pi/2\) and for each of the given material properties, while there exists only one singularity for \(\phi = \pi/4\).

<table>
<thead>
<tr>
<th>Dundurs parameters</th>
<th>(\phi = 45^\circ)</th>
<th>(\phi = 90^\circ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha = 1/2, \beta = 1/16)</td>
<td>(-0.4239)</td>
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<td></td>
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<td></td>
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<td>1.3282</td>
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<td>(\alpha = -1/2, \beta = -1/16)</td>
<td>(-0.2099)</td>
<td>(-0.4152)</td>
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<tr>
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<td></td>
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<td>1.3282</td>
</tr>
<tr>
<td></td>
<td>(-0.4152)</td>
<td>0.8902</td>
</tr>
</tbody>
</table>

The finite element discretization for the left-half of the body is shown in Fig. 6, wherein a typical banded domain of integration for computing the two-state \(M\)-integral via Eq. (8) is shown together with that for the \(J\)-integral of the membrane model. The package code ABAQUS is used for finite element solution, and the isoparametric plane strain elements with eight nodes are employed for the finite element model. It turns out that the two-state \(M\)-integral as well as the \(J\)-integral remains invariant with respect to the choice of the banded domain of integration. This suggests the robustness of the present computational scheme. A good convergence of a finite element solution has been attained with the discretization shown in Fig. 6. The implementation of the solution scheme is straightforward in that we can use a displacement-based FEM code without introducing a special procedure, as in a singular hybrid FEM (Tong et al., 1973).

We have demonstrated the application of the computational scheme discussed in Section 3 for obtaining the leading (singular) terms in an asymptotic solution for a re-entrant edge of a thin-film segment. However, we emphasize that the present scheme is also equally applicable for finding the higher-order terms in an asymptotic solution.

5. Concluding remarks

The two-state \(M\)-integral has been applied for computing the intensities of the singular near-field around the vertex of a generic composite wedge. Together with
Fig. 6. Finite element discretization for a thin film segment rigidly bonded to a substrate.

The detail of mesh discretization near the vertex
(the hatched area represents the domain for the domain integral of the $M$).

The entire finite element model
(the hatched area represents the domain for the domain integral of the $J$).

Fig. 6. Finite element discretization for a thin film segment rigidly bonded to a substrate.
the path independence of the $M$-integral, the existence of the complementary eigenfunction in the $M$-integral sense is crucial for the successful application of the present computational scheme. This existence has been proved with the aid of the eigenfunction expansion and the energetics argument associated with the $M$-integral. This proof clarifies that the weight function and the complementary eigenfield for the class of wedges under present consideration is given in the form of an eigenfunction with the complementary eigenvalue. The existence of this auxiliary field substantially simplifies the solution procedure, as demonstrated by the re-entrant edge of a thin-film segment. The advantages of the present method over the conventional schemes like singular hybrid finite element methods include a straightforward implementation by the use of a displacement-based FEM. Furthermore, the present scheme is valid for finding the higher-order terms as well as the leading (singular) terms in the eigenfunction expansion. This will be treated in detail in a subsequent work in relation to characterizing the small scaling yielding of a fracture specimen.

Acknowledgements

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Appendix A

Taking an appropriate Williams-type eigenfunction for the Airy stress function $\phi(z, \bar{z})$ (see Im, 1989), where $z = x + iy$ and $\bar{z} = x - iy$, we may write $\phi(z, \bar{z})$ as

$$
\phi(z, \bar{z}) = \text{Re} \sum_{\delta_n} \beta_n \left[ C_{1n} z^{\delta_n + 2} \frac{z^{\delta_n + 2}}{(\delta_n + 1)(\delta_n + 2)} + C_{2n} \frac{z^{\delta_n + 1}}{\delta_n + 1} + C_{3n} \frac{z^{\delta_n + 2}}{(\delta_n + 1)(\delta_n + 2)} + C_{4n} \frac{z^{\delta_n + 1}}{\delta_n + 1} \right],
$$

where $C_{kn} (k = 1 \sim 4)$ is the complex eigenvector corresponding to $\delta_n$. It is most convenient to choose the normalization for $C_{kn}$ such that $C_{(k+2)n} = C_{kn}$ for real $\delta_n$, so that the associated scaling load parameter $\beta_n$ becomes real when $\delta_n$ is real. Substituting the above expression into the relation:


\[ \sigma_{11} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x \partial y}, \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial x^2}, \]

we can obtain the stress components of Eq. (5) in an eigenfunction series. We then use the stress–strain relation to obtain the strain components, the integration of which yields the displacement components in Eq. (5). In passing, we add that one could rely upon two analytic functions to obtain the expression (5), as shown in Williams (1956).

The near field conditions for determining the eigenvalue \( \delta_n \) and the eigenvector \( C_{kn} \) in Eq. (5) are given along each interface of the two neighboring sectors:

\[
\begin{align*}
\sigma^{(k)}_{\theta \theta} &= \sigma^{(k+1)}_{\theta \theta}, \quad \sigma^{(k)}_{r \theta} = \sigma^{(k+1)}_{r \theta} \quad \text{along } \theta = \phi^{(k)}, \\
\dot{u}^{(k)}_r &= \dot{u}^{(k+1)}_r, \quad \dot{u}^{(k)}_\theta = \dot{u}^{(k+1)}_\theta \quad \text{along } \theta = \phi^{(k)},
\end{align*}
\]

where \( k = 1 \sim N \) for \( N \) sectors (see Fig. 1) of wedges joined together, and the first sector is meant by \( k = 1 \) for \( k = N \), i.e. for the continuity between the first sector and the last sector. Note that for the generic wedge composed of \( N \) sectors, there are a total \( 4N \) homogeneous near-field conditions, which are sufficient for determining an eigenvalue \( \delta_n \) and the corresponding \( 4N \) eigenvectors \( C_{mn} \) \((k = 1 \sim 4, \text{ and } m = 1 \sim N)\) through a proper normalization.

**Appendix B**

Here we derive the function \( F(\delta_j, \delta_k, \theta) \) involved in the expression for the two-state \( M \)-integral of the elastic states, which correspond to the eigenvalues \( \delta_j \) and \( \delta_k \), respectively. Since we have excluded the possibility of \( \delta_j = -1 \) and \( \delta_k = -1 \) in Section 3, the power-type eigenfunction in Eq. (5) is straightforwardly integrated without any special attention to the case \( \delta_n = -1 \). Calculating the stress and displacement components corresponding to \( \delta_j \) and \( \delta_k \) explicitly from Eq. (5), and substituting those into the expression (4b), we ultimately are led to the following:

\[
M^{(\delta_j, \delta_k)} = \int \text{Re}[\rho^{(\delta_j + \delta_k + 2)} \beta_f F(\delta_j, \delta_k, \theta) + \rho^{(\delta_j + \delta_k + 2)} \beta_f F(\delta_j, \delta_k, \theta)] \, d\theta,
\]

where

\[
F(\delta_j, \delta_k, \theta) = \frac{2(1 - \nu)}{\mu} \left[ C_1(\delta_j)C_2(\delta_k) \text{e}^{(\delta_j + \delta_k + 2)\theta} + C_2(\delta_j)C_2(\delta_k) \text{e}^{(\delta_j + \delta_k + 2)\theta} + C_3(\delta_j)C_4(\delta_k) \text{e}^{-(\delta_j + \delta_k + 2)\theta} + C_4(\delta_j)C_3(\delta_k) \text{e}^{-(\delta_j + \delta_k + 2)\theta} + C_4(\delta_j)C_3(\delta_k) \text{e}^{-(\delta_j + \delta_k + 2)\theta} + C_4(\delta_j)C_3(\delta_k) \text{e}^{(-\delta_j + \delta_k + 2)\theta} + C_4(\delta_j)C_3(\delta_k) \text{e}^{(-\delta_j + \delta_k + 2)\theta} \right].
\]
where the superposed bar ‘\(^{-}\)’ indicates the complex conjugate, and \(C_{mj} = C_m(\delta_j)\) has been used for clarity. Note that the integral (B1) disappears either for \(\text{Re}(\delta_j + \delta_k + 2) > 0\) or for \(\text{Re}(\delta_j + \delta_k + 2) < 0\). For \(\delta_j + \delta_k = -2\), the two terms inside the integrand in Eq. (B1) cease to be \(r\)-dependent for real eigenvalues, but the second term, involving the power \(\delta_j + \delta_k + 2 = 2i \text{Im}[\delta_j] = -2i \text{Im}[\delta_k]\), appears to be \(r\)-dependent for a complex \(\delta_j\). However, the argument of path-independence for \(M(\delta_j, \delta_k)\) shows that this term should disappear. Indeed, the substitution of the near-field condition (A1) into \(F(\delta_j, \delta_k, \theta)\) leads to \(F(\delta_j, \delta_k, \theta) = 0\) for \(\delta_j + \delta_k = -2\), so that the integrand turns out to be independent of \(r\) for \(\delta_j + \delta_k = -2\). Then this leads to the expression (7) for \(\delta_j = \delta_l\) and \(\delta_k = \delta_c\) by taking

\[ \beta_l = a_l - ib_l \quad \text{and} \quad \beta_c^* = e^{i\phi}. \]

References


